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EFFECT OF PORE FLUID COMPRESSIBILITY ON LOCALIZATION IN ELASTIC–PLASTIC POROUS SOLIDS UNDER UNDRAINED CONDITIONS

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Abstract—Conditions are derived for band-shaped localization, i.e. for the existence of discontinuous bifurcations of the incremental strain field, in elastic–plastic porous materials, under undrained conditions. It is assumed that the pores are filled with nearly incompressible liquid/gas mixture, and the significant parameter is the bulk modulus of the pore fluid. The extreme situation is complete incompressibility, which would correspond to full saturation of a liquid with an infinite bulk modulus. In this case the surprising result is obtained that, for plane strain and cylindrical stress states, the critical bifurcation direction is always 45° to a principal stress axis regardless of the adopted yield criterion. It is also shown in the paper that incompressibility usually (although not generally) has a stabilizing effect. For example, when the flow rule is associated, the critical hardening modulus is always non-positive. A discontinuity in strain rate is always accompanied by discontinuous pore pressure rate across the characteristic surface, and the magnitude of this jump is related to the strength of the displacement jump. One restrictive assumption used in the analysis is that the gradient of the yield criterion and the flow direction are coaxial.

A numerical evaluation of the critical bifurcation direction and the corresponding hardening modulus is given for the Mohr–Coulomb criterion, which is representative for the behavior of cohesionless granular materials such as soil.

INTRODUCTION

Discontinuous bifurcations of incremental solutions are frequently considered as a precursor of localization phenomena (such as shear bands), which have for many decades attracted considerable attention. Among the wealth of literature on this subject, we mention the classical works by Mandel (1964), Rudnicki and Rice (1975), Rice (1976), and Rice and Rudnicki (1980). Some recent results for quite general classes of elastic–plastic solids are given by Ottosen and Runesson (1991), Runesson *et al.* (1991) and Benallal (1992), to mention a few. The theoretical developments regarding valid bifurcation conditions have recently been pursued in parallel (and inspired by) achievements in developing finite element algorithms to capture localization phenomena.

Most of the work that has been published so far, including those references listed above, is related to the behavior of one-phase materials. However, localization phenomena are relevant also for elasto-plastic porous solids, whose pores are filled with a fluid (such as water, oil, etc.). More specifically, we shall in this paper consider such two-phase systems where net inflow or outflow is prevented, which is the situation that is often termed "undrained" in the soil mechanics literature. Subsequently, we adopt the framework of mixture theory, Bowen (1972), whereby the solid phase becomes synonymous with the solid skeleton. The no drainage assumption thus corresponds to the condition that the rate of volume change of the porous skeleton equals the rate of volume change of pore fluid. We should remark that, according to mixture theory, the porous solid becomes incompressible in the extreme case of incompressible pore fluid. In fact, undrained behavior is quite often in the soil mechanics literature taken to be synonymous with pointwise incompressibility. This is usually a valid approximation since, typically, the bulk modulus of water is nearly two or three orders of magnitude larger than the bulk modulus for the dry soil skeleton.

Strain localization under undrained behavior has been discussed by, for instance, Rice (1975), Rice and Cleary (1976), Rudnicki (1983) and Han and Vardoulakis (1991). We emphasize that Rudnicki (1983) discusses the relevant localization condition for the more general situation of partly drained behavior (due to finite permeability). Some general results regarding bifurcations and stability for the case of incompressibility, but without reference to localized solutions, were obtained by Runesson *et al.* (1992). In this paper, we give explicit bifurcation results for undrained behavior, and they are shown to generalize those obtained by Runesson *et al.* (1991). The porous body is assumed to be elastic–plastic, subjected to small deformations under plane strain conditions and obeying a quite general non-associated flow rule. As a consequence of the assumption of small deformations, the nominal time rate is used instead of any objective rate measure.

ELASTIC–PLASTIC RELATIONS FOR A SATURATED POROUS SOLID UNDER UNDRAINED CONDITIONS

We introduce the effective stress principle (as defined in soil mechanics), whereby the total stress s_{ij}, defined positive in tension, is decomposed into the "effective" stress σ_{ij} and the fluid stress—$p\delta_{ij}$, i.e.

$$s_{ij} = \sigma_{ij} - p\delta_{ij} \tag{1}$$

where p is the pore fluid pressure and δ_{ij} is the Kronecker delta. According to mixture theory, eqn (1) merely represents a trivial subdivision into partial stresses. It is assumed that σ_{ij} is responsible for deformation in the skeleton of the porous solid, whereas p is responsible for compression of the pore fluid. These stresses must, therefore, be defined via constitutive assumptions.

Under undrained conditions, it follows that $\dot{\varepsilon}_v$ may be related to $\dot{p}$ via the following (nonlinear) constitutive law:

$$\dot{\varepsilon}_v = -\frac{1}{K^{\mathrm{f}}}\dot{p} \quad \text{or} \quad \dot{p} = -K^{\mathrm{f}}\dot{\varepsilon}_v = -K^{\mathrm{f}}\delta_{ij}\dot{\varepsilon}_{ij} \tag{2}$$

where $\dot{\varepsilon}_{ij}$ is the strain rate tensor, $\dot{\varepsilon}_v$ is the volumetric strain rate, and K^{f} is the compression modulus of the pore fluid.

We should at this point remark, that it is generally accepted in the soil mechanics literature that the effective stress principle in eqn (1) is valid even when the porous solid skeleton is only nearly saturated, i.e. when the degree of saturation S (defined as the ratio between the pore liquid volume and total pore volume) is in the range $0.85 \leqslant S \leqslant 1.0$. In this case K^{f} should be interpreted as the bulk modulus of the two-phase mixture comprising liquid/air in the pores, and it may, typically, be two or three orders of magnitude smaller than that observed in the case of full saturation.

If the yield criterion is expressed in terms of σ_{ij} and a set of suitable hardening/softening internal variables, it is possible to express the usual tangent stiffness modulus relationship between $\dot{\sigma}_{ij}$ and $\dot{\varepsilon}_{ij}$ as

$$\dot{\sigma}_{ij} = D^{\mathrm{e}}_{ijkl}\dot{\varepsilon}_{kl} \quad \text{elastic unloading} \tag{3}$$

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}_{kl} \quad \text{plastic loading} \tag{4}$$

where the "effective" elastic–plastic tangent stiffness moduli tensor D_{ijkl} is given as

$$D_{ijkl} = D^{e}_{ijkl} - \frac{1}{A} D^{e}_{ijmn} g_{mn} f_{pq} D^{e}_{pqkl}. \tag{5}$$

In eqn (5) we have introduced the gradients

$$f_{ij} = \frac{\partial F}{\partial \sigma_{ij}}, \quad g_{ij} = \frac{\partial G}{\partial \sigma_{ij}} \tag{6}$$

where F and G are the yield and plastic potential functions, respectively. We define the positive scalar A as

$$A = f_{ij} D^{e}_{ijkl} g_{kl} + H > 0 \tag{7}$$

where H is a generalized plastic modulus. Combining eqns (1), (2), (3) and (4), we obtain

$$\dot{s}_{ij} = D^{ue}_{ijkl} \dot{\varepsilon}_{kl} \quad \text{elastic unloading} \tag{8}$$

$$\dot{s}_{ij} = D^{u}_{ijkl} \dot{\varepsilon}_{kl} \quad \text{plastic loading} \tag{9}$$

where the tangent stiffness moduli pertinent to undrained conditions (indicated by superscript "u") are defined as

$$D^{ue}_{ijkl} = D^{e}_{ijkl} + K^{f} \delta_{ij} \delta_{kl} \tag{10}$$

$$D^{u}_{ijkl} = D_{ijkl} + K^{f} \delta_{ij} \delta_{kl}. \tag{11}$$

Plastic loading occurs when

$$F = 0 \quad \text{and} \quad f_{ij} D^{e}_{ijkl} \dot{\varepsilon}_{kl} > 0. \tag{12}$$

It follows from eqns (10) and (11) that the tangent stiffness moduli tensor exists only when the fluid compressibility is finite, i.e. when $K^{f} < \infty$.

DISCONTINUOUS BIFURCATION UNDER UNDRAINED CONDITION

General results

Pursuing the standard approach, we assume that bifurcation in the incremental fields (strain rate and pore pressure rate) can occur across a characteristic internal surface C with unit normal vector n_i. We assume that plastic loading is maintained for the primary as well as the bifurcated solutions. Upon using the conditions that the rate of total traction is continuous and that mass conservation prevails across C, we obtain from eqns (1) and (2) the pertinent homogeneous equations.

$$Q_{ij} \dot{c}_j - [\dot{p}] n_i = 0 \tag{13}$$

$$\dot{c}_i n_i + \frac{1}{K^{f}} [\dot{p}] = 0 \tag{14}$$

where $\dot{c}_i$ represents the strength of discontinuity of the strain rate, whereas $[\dot{p}]$ is the discontinuity of the pore pressure rate. We have introduced the "effective" acoustic tensor Q_{ij} as

$$Q_{ij} = Q^{e}_{ij} - \frac{1}{A} b_i a_j, \quad Q^{e}_{ij} = n_k D^{e}_{iklj} n_l \tag{15}$$

where the vectors a_i and b_i are defined as

$$a_i = f_{kl} D^{e}_{klij} n_j, \quad b_i = g_{kl} D^{e}_{klij} n_j. \tag{16}$$

Clearly, bifurcation can exist only if eqns (13) and (14) possess non-trivial solutions.

Upon eliminating $[\dot{p}]$ between eqns (13) and (14), we obtain the single homogeneous equation for $\dot{c}_i$:

$$Q^{u}_{ij} \dot{c}_i = 0 \tag{17}$$

where Q^{u}_{ij} is the "undrained" acoustic tensor

$$Q^{u}_{ij} = n_k D^{u}_{ijkl} n_l = Q_{ij} + K^{f} n_i n_j. \tag{18}$$

It appears that Q^{u}_{ij} is simply obtained by augmenting Q_{ij} with a rank-one tensor. Moreover, it is clear that Q^{u}_{ij} does not exist in the case of complete incompressibility (when $K^{f} = \infty$), which is defined by the constraint equation $\dot{c}_i n_i = 0$.

As an alternative to the derivation above, we may eliminate $\dot{c}_i$ between eqns (13) and (14) to obtain the scalar equation

$$\left(n_i P_{ij} n_j + \frac{1}{K^{f}} \right) [\dot{p}] = 0 \tag{19}$$

where P_{ij} is the inverse of Q_{ij}, that is explicitly obtained as

$$P_{ij} = Q^{-1}_{ij} = P^{e}_{ij} + \frac{1}{A - a_p P^{e}_{pq} b_q} P^{e}_{ik} b_k a_l P^{e}_{lj}. \tag{20}$$

It is concluded that a non-trivial solution $[\dot{p}]$ of eqn (19) is possible only if the scalar coefficient vanishes, i.e.

$$n_i P_{ij} n_j + \frac{1}{K^{f}} = 0. \tag{21}$$

Upon introducing eqn (20), we obtain from eqn (21) a condition on H that must be satisfied at the onset of localization:

$$H = -f_{ij} D^{e}_{ijkl} g_{kl} + a_i P^{e}_{ij} b_j - \psi \frac{(a_i P^{e}_{ij} n_j)(b_k P^{e}_{kl} n_l)}{n_p P^{e}_{pq} n_q} \tag{22}$$

where the scalar ψ is given as

$$\psi = \frac{K^{f} n_i P^{e}_{ij} n_j}{1 + K^{f} n_p P^{e}_{pq} n_q}. \tag{23}$$

If we compare the expression for H in eqn (22) with that given in eqn (25) of Ottosen and Runeson (1991), we note that eqn (22) is obtained by augmenting the result for the one-phase material with an additional term due to $\psi \neq 0$. It is, therefore, of particular interest to investigate the influence the value of ψ may have on the critical bifurcation directions $n_i = n^{cr}_i$, which correspond to a maximum value of H (that is subsequently denoted H^{b}).

With the expression in eqn (22), it is possible to obtain $\dot{c}_i$ from eqn (13) as

$$\dot{c}_i = P_{ij}n_j[\dot{p}] = P^e_{ij}w_j[\dot{p}], \quad w_i = n_i - \frac{1+K^f n_p P^e_{pq} n_q}{K^f b_p P^e_{pq} n_q} b_i. \tag{24}$$

We may, as an alternative, obtain the condition (22) from a spectral analysis of the "undrained" acoustic tensor Q^u_{ij} in eqn (18). To this end, it is useful to consider the (right) eigenvalue problem

$$Q^u_{ij}z^{(k)}_j = \lambda^{(k)}Q^e_{ij}z^{(k)}_j, \quad k = 1,2,3 \tag{25}$$

where $\lambda^{(k)}$ are eigenvalues corresponding to the eigenvectors $z^{(k)}_j$. These eigenvalues and eigenvectors can be obtained quite explicitly, as shown in the Appendix. In particular, it can be shown that singularity is equivalent to the condition (22).

Isotropic elasticity

At this point we restrict our development to isotropic elasticity, which is defined by

$$D^e_{ijkl} = 2G\left(\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) + \frac{v}{1-2v}\delta_{ij}\delta_{kl}\right). \tag{26}$$

Upon inserting eqn (26) into eqn (15), we obtain expression for Q^e_{ij} and its inverse P^e_{ij}

$$Q^e_{ij} = G\left(\frac{v}{1-2v}n_in_j+\delta_{ij}\right), \quad P^e_{ij} = \frac{1}{G}\left(-\frac{1}{2(1-v)}n_in_j+\delta_{ij}\right) \tag{27}$$

whereas the vectors a_i and b_i in eqn (16) become

$$a_i = 2G\left(f_{ij}n_j + \frac{v}{1-2v}f_{r}n_i\right), \quad b_i = 2G\left(g_{ij}n_j + \frac{v}{1-2v}g_{r}n_i\right). \tag{28}$$

Inserting these expressions in eqns (22) and (23), we obtain

$$\frac{H}{2G} = 2n_if_{ik}g_{kj}n_j + \frac{v(1-\psi)}{1-v}(f_rn_ig_{ij}n_j+g_rn_if_{ij}n_j) - \frac{1+(1-2v)\psi}{1-v}(n_if_{ij}n_j)(n_kg_{kl}n_l) - \frac{v(1-2v)+v^2\psi}{(1-v)(1-2v)}f_rg_r - f_{ij}g_{ij} \tag{29}$$

where ψ is given as

$$\psi = \left(\frac{1-2v}{1-v}\frac{K^f}{2G}\right)\left(1+\frac{1-2v}{1-v}\frac{K^f}{2G}\right)^{-1}. \tag{30}$$

It is noted that ψ is independent of n_i in the case of isotropic elasticity. For soils the ratio $K^f/2G$ may, in practice, range from 10^{-1} to 10^3, which represent extreme states for moduli for partial and full liquid (water) saturation, respectively. With the choice $v = 0.2$, which is typical for clay, we observe that ψ ranges from 0.07 to 1.00. It is thus clear that complete incompressibility, which is defined by $\psi = 1(K^f = \infty)$, is indeed a relevant approximation in the case of full saturation. On the other hand, the behavior of a partially saturated soil may be quite close to what is pertinent to a dry material, for which $\psi = 0(K^f = 0)$. In conclusion, it is of interest to consider the entire interval $0 \leqslant \psi \leqslant 1$.

CRITICAL BIFURCATION DIRECTIONS AND HARDENING MODULUS

Analysis of plane strain condition

We shall first make the important assumptions that f_{ij} and g_{ij} possess the same principal directions. Moreover, we restrict our analysis to the case of plane strain, whereby we assume that two of the principal directions (x_1 and x_2) are located in the plane of interest. Hence, the x_3-direction is out-of-plane and $n_3 = 0$. Without loss of generality we also label the in-plane axes so that $f_1 \geqslant f_2$. Under very mild constraints on the flow rule, this choice will also imply that $g_1 \geqslant g_2$ (which is assumed subsequently). It is clear that the magnitude of the out-of-plane components f_3 and g_3 cannot be directly related to those in-plane.

With the assumption given above, the pertinent expression at bifurcation for H in plane strain may by expressed as a special case of eqn (29) given as

$$\frac{H}{2G} = a_1 n_1^2 + a_2 n_2^2 - \frac{1+(1-2v)\psi}{1-v}(f_1 n_1^2 + f_2 n_2^2)(g_1 n_1^2 + g_2 n_2^2) - k \tag{31}$$

where a_1 and a_2 are given as

$$a_\alpha = 2f_\alpha g_\alpha + \frac{v(1-\psi)}{1-v}(f_t g_\alpha + f_\alpha g_t), \quad \alpha = 1, 2 \tag{32}$$

and k is given as

$$k = \frac{v(1-2v)+v^2\psi}{(1-v)(1-2v)} f_t g_t + f_1 g_1 + f_2 g_2 + f_3 g_3. \tag{33}$$

In order to devise the condition for n_1^2 to represent an extreme value of H, we simply differentiate eqn (31)

$$\frac{\mathrm{d}}{\mathrm{d}(n_1^2)}\left(\frac{H}{2G}\right) = \frac{1}{1-v}(c_1 - (c_1 - c_2) n_1^2) = 0 \tag{34}$$

where

$$c_1 = (f_1 - f_2)(g_1 + vg_3) + (g_1 - g_2)(f_1 + vf_3) - \psi r_1 \tag{35}$$

$$c_2 = (f_1 - f_2)(g_2 + vg_3) + (g_1 - g_2)(f_2 + vf_3) - \psi r_2 \tag{36}$$

and

$$r_1 = (f_1 - f_2)[vg_t + (1-2v)g_2] + (g_1 - g_2)[vf_t + (1-2v)f_2] \tag{37}$$

$$r_2 = (f_1 - f_2)[vg_t + (1-2v)g_1] + (g_1 - g_2)[vf_t + (1-2v)f_1]. \tag{38}$$

It is noted that

$$c_1 - c_2 = 2[1+(1-2v)\psi](f_1 - f_2)(g_1 - g_2) \geqslant 0. \tag{39}$$

Since, in view of eqn (39),

$$\frac{\mathrm{d}^2}{\mathrm{d}(n_1^2)^2}\left(\frac{H}{2G}\right) = -\frac{1}{1-v}(c_1 - c_2) \leqslant 0 \tag{40}$$

it follows that possible solutions of eqn (34) that satisfy $0 \leqslant n_1^2 \leqslant 1$ do, in fact, represent a maximum value of H.

In the following we shall only consider the case defined by $f_1 > f_2$ and $g_1 > g_2$. Equation (34) then gives the solution

$$n_1^2 = \frac{c_1}{c_1 - c_2}, \quad n_2^2 = 1 - n_1^2 = -\frac{c_2}{c_1 - c_2} \tag{41}$$

which is valid whenever $0 \leqslant n_1^2 \leqslant 1$ (or $1 \geqslant n_2^2 \geqslant 0$) corresponding to the conditions

$$c_1 \geqslant 0, \quad c_2 \leqslant 0. \tag{42}$$

If θ denotes the angle in the x_1, x_2-plane from the x_2-axis to the normal vector (n_1, n_2), then θ is defined by

$$\tan^2 \theta = \frac{n_1^2}{n_2^2} = -\frac{c_1}{c_2}. \tag{43}$$

The corresponding maximum (critical) value H^b is obtained by inserting eqn (41) into eqn (31). Although this is a rather straightforward operation, it is not trivial to obtain a simple explicit expression, except in the special cases when $\psi = 0$ and $\psi = 1$, as discussed further below.

Whenever the conditions (42) are not satisfied, the following extreme cases appear: for $c_1 \leqslant 0$ it appears that H assumes its maximum when $\theta = 0^\circ$ ($n_1 = 0, n_2^2 = 1$). Similarly, when $c_2 \geqslant 0$ it appears that H assumes its maximum when $\theta = 90^\circ$ ($n_1^2 = 1, n_2 = 0$). Even in these two situations we refrain from stating the explicit expression for the critical value H^b for a general choice of ψ.

Analysis of cylindrical stress and strain conditions

For cylindrical states of stresses and strains, it may be assumed that the constraint of axisymmetry is satisfied by the bifurcated deformation field as well. Following the arguments presented by Perić *et al.* (1992), we conclude that the expressions for the critical bifurcation directions and the corresponding hardening modulus derived for the case of plane strain are valid also in the case of bifurcations in cylindrical systems, if we only introduce the proper symmetry condition.

Conventional triaxial tests on soil specimens are normally carried out on cylindrical samples under compressive stresses, where the magnitude of the vertical (axial) stress is allowed to increase (or decrease) from an initial state of isotropic compression. These two situations are commonly characterized as conventional triaxial compression (CTC) and conventional triaxial extension (CTE) respectively, and they are defined by $\sigma_1 = \sigma_3$ (CTC) and $\sigma_2 = \sigma_3$ (CTE). In the case of axisymmetry, it is noted that index 3 denotes the circumferential direction, whereas indices 1 and 2, respectively, denote the major and minor principal stress directions in the radial plane. Moreover, it will be assumed that the principal directions of σ_{ij} are identical to those of f_{ij} and g_{ij}, which is the case (at least) when the yield criterion and plastic potential are isotropic. In the case of the CTC test, it follows that the pertinent identities $f_1 = f_3$ and $g_1 = g_3$ will affect only the value of c_1 in eqn (35). In the case of CTE, on the other hand, the identities $f_2 = f_3$ and $g_2 = g_3$ will influence only the value of c_2 in eqn (36). However, even in these cases it is not possible to obtain simple explicit expressions of H^b, except when $\psi = 0$ and $\psi = 1$.

One phase material

In order to provide a complete description we shall briefly comment on the solution that is pertinent to the one-phase material, i.e. $\psi = 0$. This situation was treated thoroughly by Runesson *et al.* (1991). We briefly remark that it is possible to achieve either of the three following situations: $c_1 \leqslant 0$, ($c_2 \leqslant 0$) corresponding to $\theta = 0^\circ$; $c_1 \geqslant 0, c_2 \leqslant 0$ corresponding to $0^\circ < \theta < 90^\circ$; and, finally ($c_1 \geqslant 0$), $c_2 \geqslant 0$ corresponding to $\theta = 90^\circ$. It is not possible to make any general conclusion about the sign of H^b in either of these situations, except when the flow rule is associated, in which case $H^b \leqslant 0$.

Incompressible pore fluid

The case of a completely incompressible pore fluid, i.e. $\psi = 1$, gives upon insertion into eqns (35) and (36)

$$c_1 = 2(1-\nu)(f_1 - f_2)(g_1 - g_2) \geqslant 0 \tag{44}$$

$$c_2 = -c_1 \leqslant 0 \tag{45}$$

and

$$c_1 - c_2 = 2c_1 \geqslant 0. \tag{46}$$

Hence, we obtain the generally valid solution from eqn (41) as

$$n_1^2 = n_2^2 = \tfrac{1}{2} \quad \text{or } \theta = \pm 45^\circ \tag{47}$$

corresponding to the critical value H^b from eqn (31)

$$\frac{H^b}{2G} = -\tfrac{1}{2}(f_1 + f_2)(g_1 + g_2) - f_3 g_3 - \frac{\nu}{1-2\nu} f_v g_v. \tag{48}$$

In the special case where the flow is associated, we obtain from eqn (48)

$$\frac{H^b}{2G} = -\tfrac{1}{2}(f_1 + f_2)^2 - f_3^2 - \frac{\nu}{1-2\nu} f_v^2 < 0. \tag{49}$$

In accordance with the previous discussion, the expressions for H^b in eqns (48) and (49) are valid, not only for plane strain, but also for cylindrical states, if only the pertinent symmetry condition ($f_1 = f_3$, $g_1 = g_3$ for CTC or $f_2 = f_3$, $g_2 = g_3$ for CTE) is imposed.

The remarkable result expressed in eqn (47), which is valid regardless of the adopted yield criterion, is important to dwell on. It is worth noting that it is the specific type of incompressibility that is imposed as an external constraint that is crucial. For example, adopting elastic and plastic incompressibility in a drained analysis [$\nu = 1/2$ and $\beta = 0$ in the pressure dependent model used by Rudnicki and Rice (1975)], does not result in the 45° angle. [This fact was pointed out to us by Rudnicki (1994).]

As to experimental evidence, Han and Vardoulakis (1991) report the orientation of the shear band to be in the range between 57° to 62° towards the horizontal plane for undrained plane strain compression tests. It should be noted that these angles were measured at the end of the test, which corresponds to very large deformation and significant geometric distortion. At this late stage the behavior of the shear band and its vicinity reflects partial drainage and changed boundary conditions, which results in an orientation that is different from 45°. It is emphasised that our analytical result refers strictly to onset of bifurcation, i.e. the initial stage. Another possible reason for discrepancy between observed directions and the theoretical result is that the critical modulus H^b is not actually attained. This may be explained as follows: the analysis presented above assumes a smooth stress–strain behavior in such a way that the critical value H^b is attained somewhere along the loading path. However, this may not be the case. For example, for semi-brittle response, the critical state may be traversed right at the peak stress and $H < H^b$ right after peak, where H^b is given in eqn (48). In this situation two bifurcation directions $\theta^{(1)}$ and $\theta^{(2)}$ are possible, and these are obtained form eqn (31) with $\psi = 1$:

$$\tan^2\theta^{(1)} = \frac{1+\sqrt{\dfrac{H^{\mathrm{b}}-H}{G(f_1-f_2)(g_1-g_2)}}}{1-\sqrt{\dfrac{H^{\mathrm{b}}-H}{G(f_1-f_2)(g_1-g_2)}}}, \quad \theta^{(2)} = 90^\circ - \theta^{(1)}. \tag{50}$$

Apparently, two real solutions are possible as long as the condition

$$H^{\mathrm{b}} - G(f_1-f_2)(g_1-g_2) < H < H^{\mathrm{b}} \tag{51}$$

is satisfied. The (unique) solution $\theta = \pm 45^\circ$ is obtained when $H = H^{\mathrm{b}}$, whereas $\theta = 90^\circ$, when H assumes the lower limit in eqn (51). For smaller values of H, no bifurcation solution is possible.

BIFURCATION CONDITION FOR THE MOHR-COULOMB YIELD CRITERION

The yield criterion and plastic potential according to Mohr–Coulomb can be defined by

$$F = \tfrac{1}{2}(\sigma_{\mathrm{I}}-\sigma_{\mathrm{III}})+\tfrac{1}{2}(\sigma_{\mathrm{I}}+\sigma_{\mathrm{III}})\sin\phi - c = 0 \tag{52}$$

$$G = \tfrac{1}{2}(\sigma_{\mathrm{I}}-\sigma_{\mathrm{III}})+\tfrac{1}{2}(\sigma_{\mathrm{I}}+\sigma_{\mathrm{III}})\sin\phi^* \tag{53}$$

where $\sigma_{\mathrm{I}} \geqslant \sigma_{\mathrm{II}} \geqslant \sigma_{\mathrm{III}}$ are the principal stresses (which are taken positive in tension), ϕ is the angle of internal friction, ϕ^* is the angle of dilatancy, and c is a cohesion intercept.

We recall that the principal stresses located in-plane are ordered such that $\sigma_1 \geqslant \sigma_2$. However, three cases are distinguished depending on which stress is identified with the intermediate principal stress σ_{II}.

Case A. Consider the situation defined by $\sigma_1-\sigma_3 \geqslant 0$ and $\sigma_2-\sigma_3 \geqslant 0$. Since $\sigma_1 \geqslant \sigma_2$ (by definition), we obtain $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$, and hence $\sigma_{\mathrm{I}} = \sigma_1$ whereas $\sigma_{\mathrm{III}} = \sigma_3$. We obtain from eqns (52) and (53):

$$f_1 = \tfrac{1}{2}(1+\sin\phi), \quad f_2 = 0, \quad f_3 = -\tfrac{1}{2}(1-\sin\phi) \tag{54}$$

$$g_1 = \tfrac{1}{2}(1+\sin\phi^*), \quad g_2 = 0, \quad g_3 = -\tfrac{1}{2}(1-\sin\phi^*) \tag{55}$$

and

$$f_v = \sin\phi, \quad g_v = \sin\phi^*. \tag{56}$$

The character of the solution is defined by the pertinent values of c_1 and c_2, which are given from eqns (35) and (36) as

$$c_1 = \tfrac{1}{2}[1-\nu+\sin\phi+\sin\phi^*+(1+\nu)\sin\phi\sin\phi^*] - \frac{\nu\psi}{2}[\sin\phi+\sin\phi^*+2\sin\phi\sin\phi^*] \tag{57}$$

$$c_2 = -\frac{\nu}{2}(1-\sin\phi\sin\phi^*) - \frac{\psi}{2}[1-2\nu+(1-\nu)(\sin\phi+\sin\phi^*)+\sin\phi\sin\phi^*]. \tag{58}$$

In the following it will be assumed that $\phi \geqslant \phi^* \geqslant 0$. Since $0 \leqslant \psi \leqslant 1$ and $-1 < \nu < 1/2$, it follows from eqns (57) and (58) quite simply that $c_1 \geqslant 0$ and $c_2 \leqslant 0$, i.e. the solution given by eqn (43) is the appropriate one.

When $\psi = 0$, we retrieve the solution obtained by Runesson *et al.* (1991):

$$\tan^2\theta = \frac{1-\nu+\sin\phi+\sin\phi^*+(1+\nu)\sin\phi\sin\phi^*}{\nu(1-\sin\phi\sin\phi^*)} \tag{59}$$

$$\frac{H^{\mathrm{b}}}{2G} = \frac{\nu^2(1-\sin\phi\sin\phi^*)^2}{4(1-\nu)(1+\sin\phi)(1+\sin\phi^*)} - \frac{(1-\sin\phi)(1-\sin\phi^*)}{4(1-\nu)}. \tag{60}$$

When $\psi = 1$ we obtain from eqn (48) that the critical directions $\theta = \pm 45^\circ$ correspond to

$$\frac{H^{\mathrm{b}}}{2G} = -\frac{1}{2(1-2\nu)}\sin\phi\sin\phi^* + \tfrac{1}{8}(1+\sin\phi)(1+\sin\phi^*) - \tfrac{1}{2} \leqslant 0. \tag{61}$$

It is observed that, whereas nothing can be said in general about the sign of H^{b} in eqn (60), we obtain $H^{\mathrm{b}} \leqslant 0$ in eqn (61) for the case of incompressibility.

Case B. Consider the situation where $\sigma_1 - \sigma_3 \geqslant 0$ and $\sigma_2 - \sigma_3 \leqslant 0$. Combining these inequalities, we conclude that $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$, and hence $\sigma_{\mathrm{I}} = \sigma_1$ whereas $\sigma_{\mathrm{III}} = \sigma_2$. We then obtain from eqns (52) and (53):

$$f_1 = \tfrac{1}{2}(1+\sin\phi), \quad f_2 = -\tfrac{1}{2}(1-\sin\phi), \quad f_3 = 0 \tag{62}$$

$$g_1 = \tfrac{1}{2}(1+\sin\phi^*), \quad g_2 = -\tfrac{1}{2}(1-\sin\phi^*), \quad g_3 = 0 \tag{63}$$

and

$$f_v = \sin\phi, \quad g_v = \sin\phi^*. \tag{64}$$

We now obtain c_1 and c_2 from eqns (35) and (36) as

$$c_1 = \tfrac{1}{2}(2+\sin\phi+\sin\phi^*) - \frac{\psi}{2}(4\nu-2+\sin\phi+\sin\phi^*) \tag{65}$$

$$c_2 = -\tfrac{1}{2}(2-\sin\phi-\sin\phi^*) + \frac{\psi}{2}(4\nu-2-\sin\phi-\sin\phi^*). \tag{66}$$

Also in this case it appears that $c_1 \geqslant 0$ and $c_2 \leqslant 0$ generally holds.

When $\psi = 0$, we retrieve the solution obtained by Runesson *et al.* (1991):

$$\tan^2\theta = \frac{2+\sin\phi+\sin\phi^*}{2-\sin\phi-\sin\phi^*} \tag{67}$$

$$\frac{H^{\mathrm{b}}}{2G} = \frac{1}{16(1-\nu)}(\sin\phi-\sin\phi^*)^2 \geqslant 0. \tag{68}$$

When $\psi = 1$, we obtain from eqn (48) that the critical directions $\theta = \pm 45^\circ$ correspond to

$$\frac{H^{\mathrm{b}}}{2G} = -\frac{1}{2(1-\nu)}\sin\phi\sin\phi^* \leqslant 0. \tag{69}$$

It is interesting to note that $H^{\mathrm{b}} \geqslant 0$ for $\psi = 0$, whereas $H^{\mathrm{b}} \leqslant 0$ for $\psi = 1$, i.e. the stabilizing effect of complete incompressibility is demonstrated explicitly.

Since we have assumed that $0 \leqslant \phi^* \leqslant \phi$, we obtain for $\psi = 0$ the following bounds:

$$0 \leqslant \frac{H^b}{2G} \leqslant \frac{1}{16(1-\nu)} \sin^2 \phi \tag{70}$$

whereas $\psi = 1$ gives

$$-\frac{1}{2(1-2\nu)} \sin^2 \phi \leqslant \frac{H}{2G} \leqslant 0. \tag{71}$$

Case C. Finally, we consider the situation defined by $\sigma_1 - \sigma_3 \leqslant 0$ and $\sigma_2 \leqslant \sigma_3 \leqslant 0$. Since $\sigma_1 \geqslant \sigma_2$ (by definition), we obtain $\sigma_3 \geqslant \sigma_1 \geqslant \sigma_2$, and hence $\sigma_1 = \sigma_3$ whereas $\sigma_{III} = \sigma_2$. We then obtain from eqns (52) and (53):

$$f_1 = 0, \quad f_2 = -\tfrac{1}{2}(1-\sin\phi), \quad f_3 = \tfrac{1}{2}(1+\sin\phi) \tag{72}$$

$$g_1 = 0, \quad g_2 = -\tfrac{1}{2}(1-\sin\phi^*), \quad g_3 = \tfrac{1}{2}(1+\sin\phi^*) \tag{73}$$

and

$$f_v = \sin\phi, \quad g_v = \sin\phi^*. \tag{74}$$

From eqns (35) and (36) we obtain

$$c_1 = \frac{\nu}{2}(1-\sin\phi\sin\phi^*) - \frac{\psi}{2}[2\nu - 1 + (1-\nu)(\sin\phi + \sin\phi^*) - \sin\phi\sin\phi^*] \tag{75}$$

$$c_2 = \tfrac{1}{2}[\nu - 1 + \sin\phi + \sin\phi^* - (1+\nu)\sin\phi\sin\phi^*] - \frac{\nu\psi}{2}(\sin\phi + \sin\phi^* - 2\sin\phi\sin\phi^*) \tag{76}$$

and it is concluded that $c_1 \geqslant 0$ and $c_2 \leqslant 0$ generally hold.

When $\psi = 0$, we retrieve the solution obtained by Runesson *et al.* (1991):

$$\tan^2\theta = \frac{\nu(1-\sin\phi\sin\phi^*)}{1-\nu+\sin\phi+\sin\phi^*+(1+\nu)\sin\phi\sin\phi^*} \tag{77}$$

$$\frac{H^b}{2G} = \frac{\nu^2(1-\sin\phi\sin\phi^*)^2}{4(1-\nu)(1-\sin\phi)(1-\sin\phi^*)} - \frac{(1+\sin\phi)(1+\sin\phi^*)}{4(1-\nu)}. \tag{78}$$

When $\psi = 1$, on the other hand, we obtain from eqn (48) that the critical directions $\theta = \pm 45^\circ$ correspond to

$$\frac{H^b}{2G} = -\frac{1}{2(1-2\nu)}\sin\phi\sin\phi^* + \tfrac{1}{8}(1-\sin\phi)(1-\sin\phi^*) - \tfrac{1}{2} \leqslant 0. \tag{79}$$

The results for the Mohr–Coulomb criterion are illustrated in Figs 1 and 2 for the particular choice $\phi = 25.4^\circ$ and $\phi^* = 2.8^\circ$. The effect of varying ψ from zero (no pore fluid) to unity (incompressible pore fluid, full saturation) is considered for different values of ν. In Fig. 1(b) we have also indicated the solution for θ_A that was suggested by Arthur *et al.* (1977):

$$\theta_A = 45^\circ + \tfrac{1}{4}(\phi + \phi^*) \tag{80}$$

which is known to be a good approximation of the value obtained in eqn (67), which is valid for $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$ when $\psi = 0$. It is noted that the critical value of θ is not affected by the value of ν when $\psi = 1$, and also, when $\psi = 0$ in the particular case where $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$.

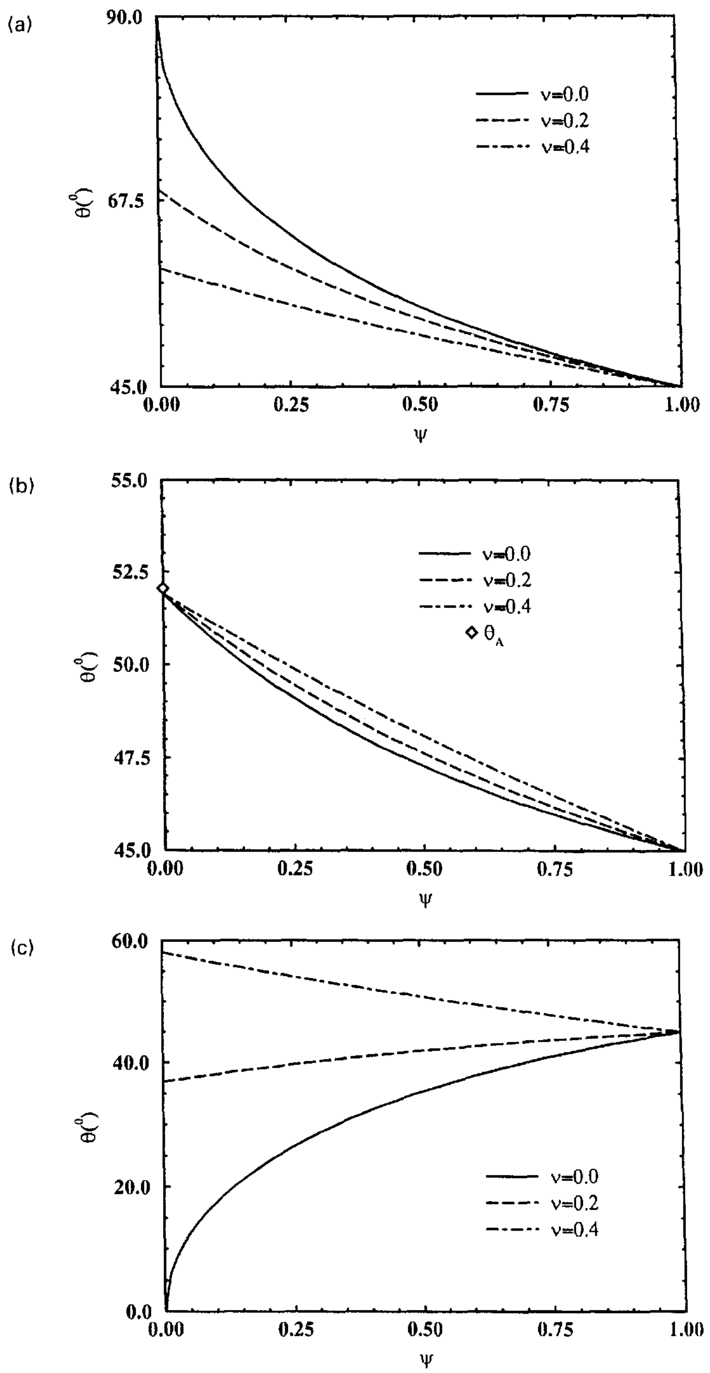
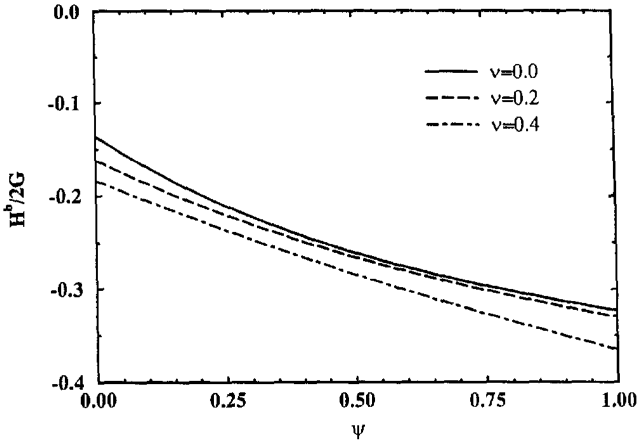
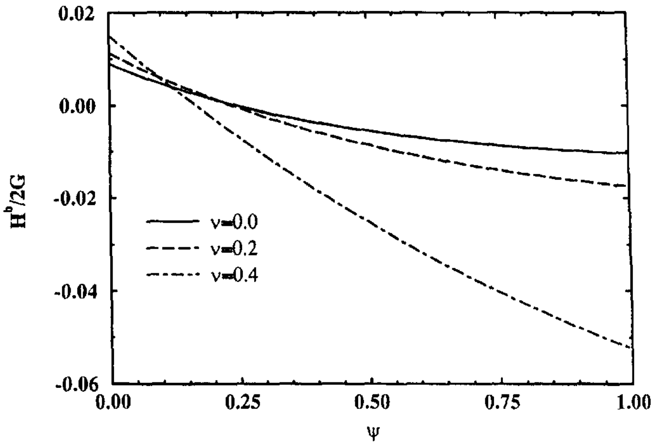


Fig. 1. Mohr–Coulomb's yield criterion. Variation of the critical value of θ with ψ for $\phi = 25.4°$ and $\phi^* = 2.8°$ in the cases (a) $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$; (b) $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$; (c) $\sigma_3 \geqslant \sigma_1 \geqslant \sigma_2$.

(a)



(b)



(c)

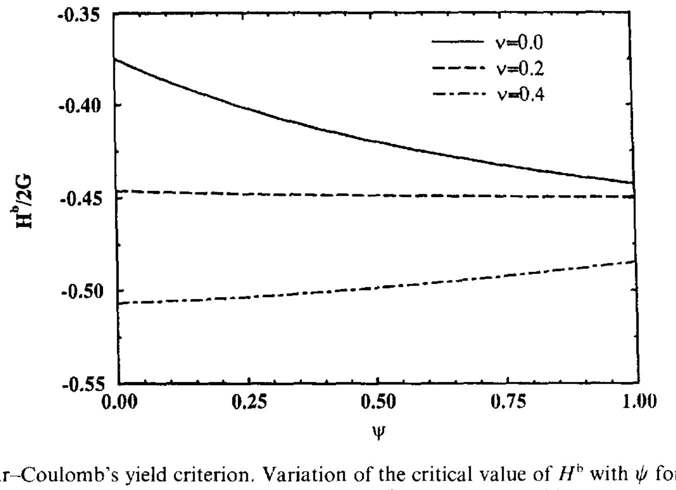


Fig. 2. Mohr–Coulomb's yield criterion. Variation of the critical value of H^b with ψ for $\phi = 25.4^\circ$ and $\phi^* = 2.8^\circ$ in the cases (a) $\sigma_1 \geq \sigma_2 \geq \sigma_3$; (b) $\sigma_1 \geq \sigma_3 \geq \sigma_2$; (c) $\sigma_3 \geq \sigma_1 \geq \sigma_2$.

Moreover, the strongly stabilizing effect of increasing ψ (in the sense that H^b decreases) is observed except in the case where $\sigma_3 \geqslant \sigma_1 \geqslant \sigma_2$ [Fig. 2(c)]. In this case H^b is even slightly increasing for large values of ν. Figures 3 and 4, finally, show the influence of non-associativity on the critical hardening modulus for the extreme situations $\psi = 0$ and $\psi = 1$ in the particular case that $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$.

Finally, we check the sensitivity of θ for a variation of H in the event of non-smooth response so that $H < H^b$ is possible. Consider Case B above and assume that $H = \varphi H^b$,

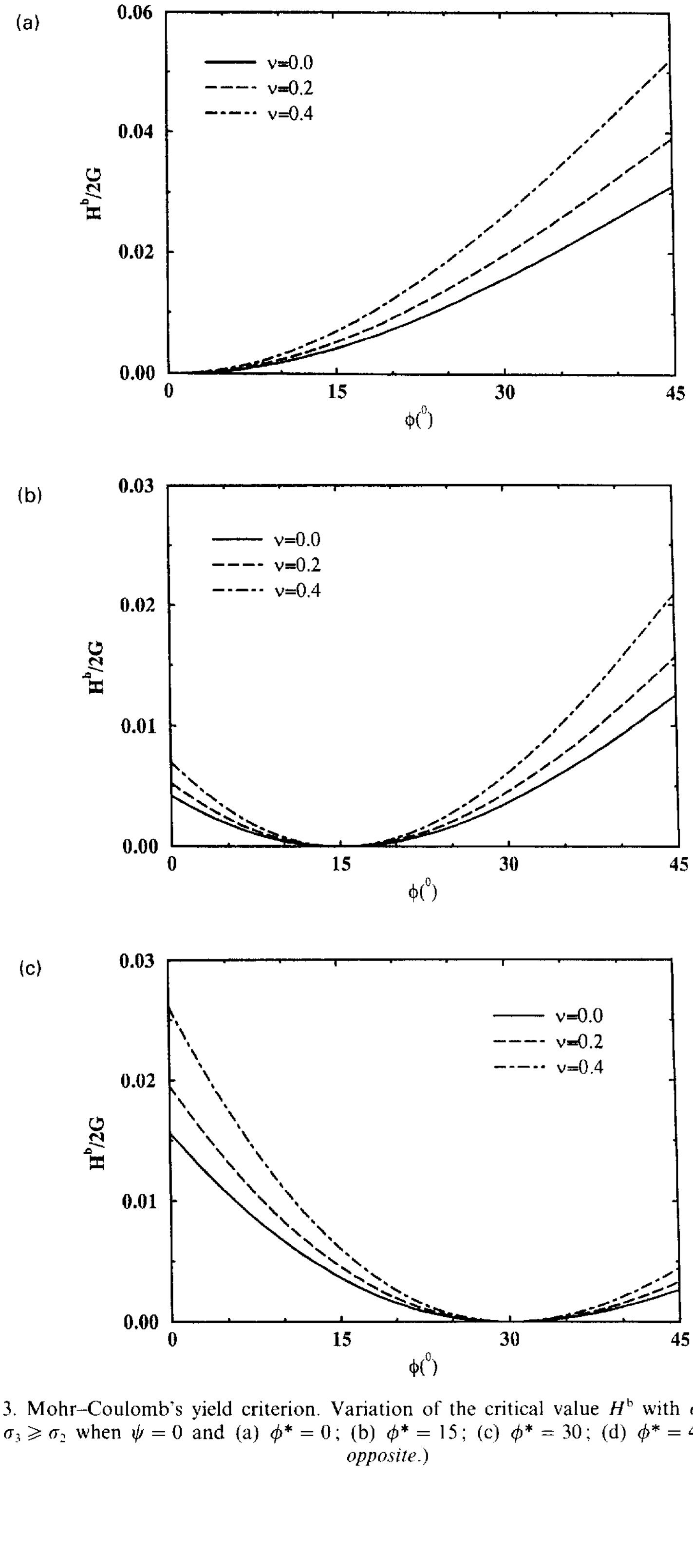


Fig. 3. Mohr–Coulomb's yield criterion. Variation of the critical value H^b with ϕ for the case $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$ when $\psi = 0$ and (a) $\phi^* = 0$; (b) $\phi^* = 15$; (c) $\phi^* = 30$; (d) $\phi^* = 45$. (*Continued opposite.*)

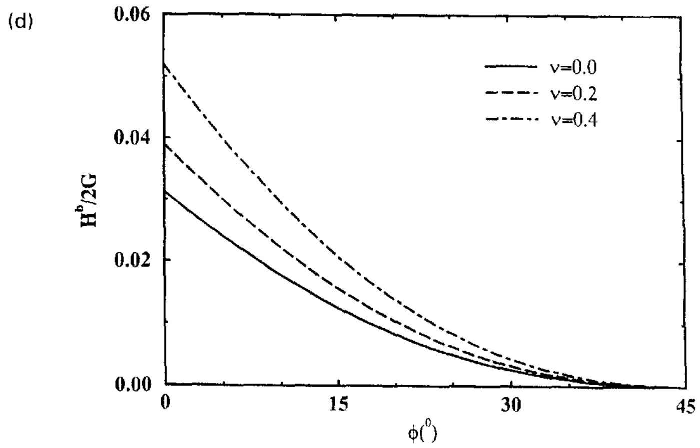


Fig. 3—*Continued.*

where the scalar φ satisfies

$$1 \leqslant \varphi \leqslant 1+\frac{1-v}{\sin\phi\sin\phi^*}. \tag{81}$$

The pertinent bifurcation directions $\theta^{(1)}$ and $\theta^{(2)}$ are then given as:

$$\tan^2\theta^{(1)} = \frac{1+\sqrt{\dfrac{\varphi-1}{1-v}\sin\phi\sin\phi^*}}{1-\sqrt{\dfrac{\varphi-1}{1-v}\sin\phi\sin\phi^*}}. \tag{82}$$

For example, $\varphi = 1.1$ and $v = 0.2$ gives $\theta^{(1)} = \pm 46.5^\circ$ and $\theta^{(2)} = \pm 43.5^\circ$.

CONCLUSIONS

The effect of pore fluid compressibility on the potential discontinuous bifurcations in elastic–plastic porous solids subjected to undrained condition was investigated. The critical bifurcation direction in the case of complete incompressibility can be obtained in the limit as calculated for a nearly incompressible pore fluid.

An important finding of the analysis is that bifurcation is, in general, delayed with increasing compression modulus of pore fluid. This finding is in agreement with Han and Vardoulakis (1991), who basically arrived at the same conclusion based on exclusively experimentally observed behavior. When the pore fluid is incompressible ($\psi = 1$), and when the flow rule is associated it was shown that the (potential for) bifurcation requires material softening. Moreover, it was shown that the critical bifurcation directions are always oriented at 45° to the principal stress axes for plane strain state, when $\psi = 1$, and this result holds regardless of the adopted yield criterion. Finally, for the Mohr–Coulomb criterion it was shown that nonassociativity, which is represented by the difference between the friction angle ϕ and the dilatancy angle ϕ^* has a significant influence on the bifurcation results.

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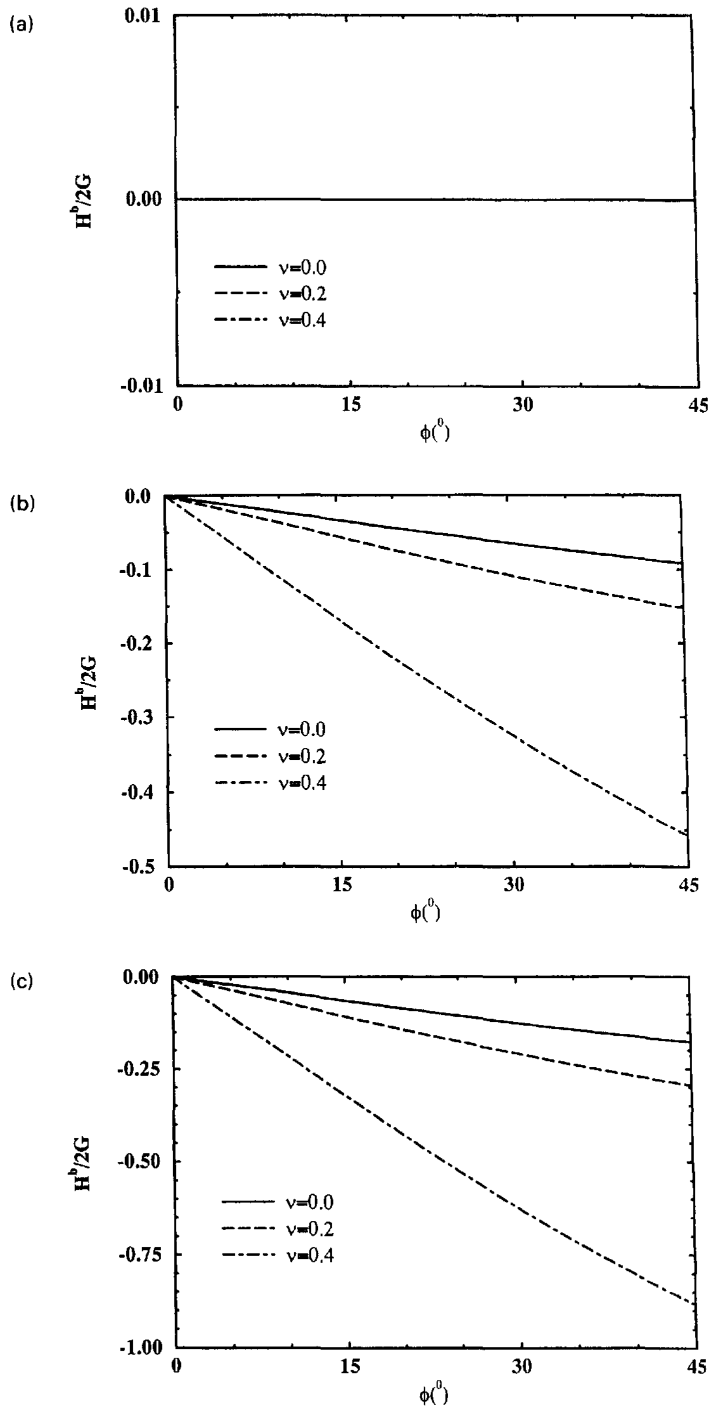

Fig. 4. Mohr–Coulomb's yield criterion. Variation of the critical value H^b with ϕ for the case $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$ when $\psi = 1$ and (a) $\phi^* = 0$; (b) $\phi^* = 15$; (c) $\phi^* = 30$; (d) $\phi^* = 45$. (*Continued opposite.*)

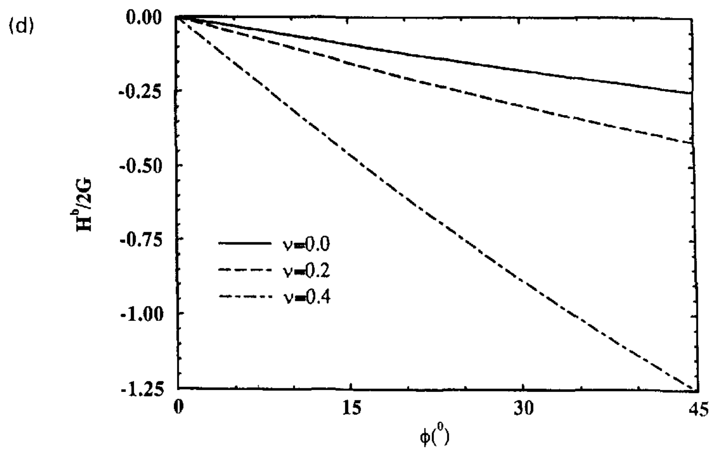

Fig. 4—*Continued.*

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APPENDIX

Spectral properties of undrained acoustic tensor

Consider the eigenvalue problem

$$Q^{u}_{ij}z^{(k)}_{j} = \lambda^{(k)}Q^{e}_{ij}z^{(k)}_{j}, \quad k = 1,2,3 \tag{A1}$$

where Q^{u}_{ij} is given as

$$Q^{u}_{ij} = Q^{e}_{ij} - \frac{1}{A}b_i a_j + K^f n_i n_j \tag{A2}$$

Upon premultiplying eqn (A1) by $(Q^e)^{-1/2}_{ki}$, we obtain the transformed problem

$$\left(\delta_{ij} - \frac{1}{A}b'_i a'_j + K^f n'_i n'_j\right)z''^{(k)}_{j} = \lambda^{(k)}z''^{(k)}_{i}, \quad k = 1,2,3 \tag{A3}$$

where

$$a_i' = (Q^e)_{ij}^{-1/2} a_j, \quad b_i' = (Q^e)_{ij}^{-1/2} b_j, \quad n_i' = (Q^e)_{ij}^{-1/2} n_j \tag{A4}$$

and

$$z_i'' = (Q^e)_{ij}^{-1/2} z_j. \tag{A5}$$

Two eigenvalues are obtained upon assuming that the corresponding eigenvectors can be expressed as

$$z_i'' = \alpha b_i' + \beta n_i' \tag{A6}$$

where α and β are constants, which are not both zero. Upon inserting eqn (A6) into eqn (A3), we obtain the homogeneous system

$$\begin{bmatrix} 1 - \frac{1}{A} a_i' b_i' - \lambda & -\frac{1}{A} a_i' n_i' \\ K' b_i' n_i' & 1 + K' n_i' n_i' - \lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0. \tag{A7}$$

A non-trivial solution of eqn (A7) requires that the determinant of the coefficient matrix vanishes, which gives the two values of λ as

$$\lambda^{(1,2)} = 1 - \frac{1}{2A} a_i' b_i' + \frac{K^f}{2} n_i' n_i' \mp \frac{1}{2} \left[\left(\frac{1}{A} a_i' b_i' + K' n_i' n_i' \right)^2 - \frac{4K^f}{A} a_i' n_i' b_j' n_j' \right]^{1/2}. \tag{A8}$$

Upon introducing the expressions in eqn (A4), we finally obtain

$$\lambda^{(1,2)} = 1 - \frac{1}{2A} a_i P_{ij}^e b_j + \frac{K^f}{2} n_i P_{ij}^e n_j \mp D \tag{A9}$$

where

$$D = \frac{1}{2} \sqrt{\left(\frac{1}{A} a_i P_{ij}^e b_j + K' n_i P_{ij}^e n_j \right)^2 - \frac{4K^f}{A} (a_i P_{ij}^e n_j)(b_k P_{kl}^e n_l)}. \tag{A10}$$

We have introduced the inverse P_{ij}^e of Q_{ij}^e. Since Q_{ij}^e is positive definite, it follows that P_{ij}^e is also positive definite (and $n_i P_{ij}^e n_j > 0$).

It appears that $\lambda^{(1)} < \lambda^{(2)}$, if the minus sign in eqn (A9) is chosen to define $\lambda^{(1)}$. The corresponding eigenvectors $z_i^{(1)}$ and $z_i^{(2)}$ are given as

$$z_i^{(\alpha)} = \gamma P_{ij}^e \left(n_j - \frac{1 + K' n_k P_{kl}^e n_l - \lambda^{(\alpha)}}{K' b_k P_{kl}^e n_l} b_j \right). \tag{A11}$$

where γ is a scalar. The third eigenvalue (which is of no interest for the present analysis) is $\lambda^{(3)} = 1$. This may be readily shown from eqn (A3) by choosing $z_i''^{(3)}$ as orthogonal to b_i' as well as to n_i'. In conclusion, $\lambda^{(1)}$ is the smallest eigenvalue and singularity of Q_{ij}^u is achieved when $\lambda^{(1)} = 0$.